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Holographic Wilson loops

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Abstract: We show that all half-BPS Wilson loop operators in $\mathcal{N} = 4$ SYM – which are labeled by Young tableaus – have a gravitational dual description in terms of $D5$-branes or alternatively in terms of $D3$-branes in $\mathrm{AdS}_5\times\mathrm{S}^5$. We prove that the insertion of a half-BPS Wilson loop operator in the $\mathcal{N} = 4$ SYM path integral is achieved by integrating out the degrees of freedom on a configuration of bulk $D5$-branes or alternatively on a configuration of bulk $D3$-branes. The bulk $D5$-brane and $D3$-brane descriptions are related by bosonization.

Keywords: Supersymmetric gauge theory, AdS-CFT Correspondence.

Contents

1. Introduction and conclusion

A necessary step in describing string theory in terms of a holographic dual gauge theory is to be able to map all gauge invariant operators of the field theory in string theory, as all physical information is captured by gauge invariant observables.

Gauge theories can be formulated in terms of a non-abelian vector potential or alternatively in terms of gauge invariant Wilson loop variables. The formulation in terms of non-abelian connections makes locality manifest while it has the disadvantage that the vector potential transforms inhomogeneously under gauge transformation and is therefore not a physical observable. The formulation in terms of Wilson loop variables makes gauge invariance manifest at the expense of a lack of locality. The Wilson loop variables, being non-local, appear to be the natural set of variables in which the bulk string theory formulation should be written down to make holography manifest. It is therefore interesting to consider the string theory realization of Wilson loop operators.[1]

[1] This has been done for Wilson loops in the fundamental representation by [1, 2]

Significant progress has been made in mapping local gauge invariant operators in gauge theory in the string theory dual. Local operators in the boundary theory correspond to bulk string fields [3 – 6]. Furthermore, the correlation function of local gauge invariant operators is obtained by evaluating the string field theory action in the bulk with prescribed sources at the boundary.

Wilson loop operators are an interesting set of non-local gauge invariant operators in gauge theory in which the theory can be formulated. Mathematically, a Wilson loop is the trace in an arbitrary representation R of the gauge group G of the holonomy matrix associated with parallel transport along a closed curve C in spacetime. Physically, the expectation value of a Wilson loop operator in some particular representation of the gauge group measures the phase associated with moving an external charged particle with charge R around a closed curve C in spacetime.

In this paper we show that all half-BPS operators in four dimensional $\mathcal{N} = 4$ SYM with gauge group $SU(N)$ – which are labeled by an irreducible representation of $SU(N)$ — can be realized in the dual gravitational description in terms of $D5$-branes or alternatively in terms of $D3$-branes in $\mathrm{AdS}_5\times\mathrm{S}^5$. We show this by explicitly integrating out the physics on the $D5$-branes or alternatively on the $D3$-branes and proving that this inserts a half-BPS Wilson loop operator in the desired representation in the $\mathcal{N} = 4$ SYM path integral.

The choice of representation of $SU(N)$ can be conveniently summarized in a Young tableau. We find that the data of the tableau can be precisely encoded in the AdS bulk description. Consider a Young tableau for a representation of $SU(N)$ with n_i boxes in the i-th row and m_j boxes in the j-th column:

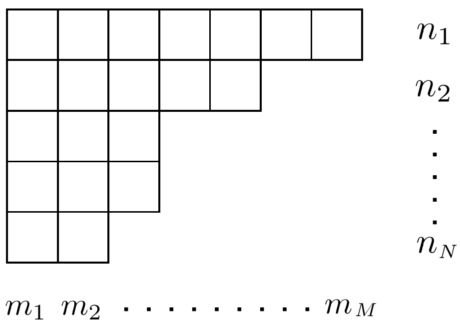

Figure 1: A Young tableau. For $SU(N)$, $i \leq N$ and $m_j \leq N$ while M and n_i are arbitrary.

We show that the Wilson operator labeled by this tableau is generated by integrating out the degrees of freedom on M coincident $D5$-branes in $\mathrm{AdS}_5\times\mathrm{S}^5$ where the j-th $D5$-brane has m_j units of fundamental string charge dissolved in it. If we label the j-th $D5$-brane carrying m_j units of charge by $D5_{m_j}$, the Young tableau in figure 1. has a bulk description in terms of a configuration of $D5$-branes given by $(D5_{m_1}, D5_{m_2}, \ldots, D5_{m_M})$.

We show that the same Wilson loop operator can also be represented in the bulk description in terms of coincident $D3$-branes[2] in $\mathrm{AdS}_5\times\mathrm{S}^5$ where the i-th $D3$-brane has n_i

[2]The number of $D3$-branes depends on the length of the first column, which can be at most N

units of fundamental string charge dissolved in it[3]. If we label the i-th $D3$-brane carrying n_i units of charge by $D3_{n_i}$, the Young tableau in figure 1. has a bulk description in terms of a configuration of $D3$-branes[4] given by $(D3_{n_1}, D3_{n_2}, \ldots, D3_{n_N})$.

The way we show that the bulk description of half-BPS Wilson loops is given by D-branes is by studying the effective field theory dynamics on the N $D3$-branes that generate the $\mathrm{AdS}_5\times\mathrm{S}^5$ background in the presence of bulk $D5$ and $D3$-branes. This effective field theory describing the coupling of the degrees of freedom on the bulk D-branes to the $\mathcal{N}=4$ SYM fields is a defect conformal field theory (see e.g. [8 − 10]). It is by integrating out the degrees of freedom associated with the bulk D-branes in the defect conformal field theory that we show the correspondence between bulk branes and Wilson loop operators. We can carry out this procedure exactly and show that this results in the insertion of a half-BPS Wilson loop operator in the $\mathcal{N}=4$ SYM theory and that the mapping between the Young tableau data and the bulk $D5$ and $D3$ brane configuration is the one we described above.

We find that the $D3$-brane description of the Wilson loop is related to the $D5$-brane description by bosonizing the localized degrees of freedom of the defect conformal field theory. The degrees of freedom localized in the codimension three defect, which corresponds to the location of the Wilson line, are fermions when the bulk brane is a $D5$-brane. We find that if we quantize these degrees of freedom as bosons instead, which is allowed in $0+1$ dimensions, that the defect conformal field theory captures correctly the physics of the bulk $D3$-branes.

One of outstanding issues in the gauge/gravity duality is to exhibit the origin of the loop equation of gauge theory in the gravitational description. This important problem has thus far remained elusive. Having shown that Wilson loops are more naturally described in the bulk by D-branes instead of by fundamental strings, it is natural to search for the origin of the loop equation of gauge theory in the D-brane picture instead of the fundamental string picture. This is an interesting problem that we hope to turn to in the future.

Having obtained the bulk description of all half-BPS Wilson loop operators in $\mathcal{N}=4$ SYM in terms of D-branes, it is natural to study the Type IIB supergravity solutions describing these Wilson loops. Precisely this program has been carried out by Lin, Lunin and Maldacena [11] for the case half-BPS local operators in $\mathcal{N}=4$ SYM. In a recent interesting paper by Yamaguchi [12], a supergravity ansatz was written down that can be used to search for these solutions. It would be interesting to solve the supergravity BPS equations for this case.

The description of Wilson loop operators in terms of a defect conformal field theory seems very economical and might be computationally useful when performing calculations

[3]This D-brane has been previously considered in the study of Wilson loops by Drukker and Fiol [7]. In this paper we show that these D-branes describe Wilson loops in a representation of the gauge group which we determine.

[4]There can be at most N $D3$-branes. A $D3$-brane with $\mathrm{AdS}_2\times\mathrm{S}^2$ worldvolume is a domain wall in AdS_5 and crossing it reduces the amount of five-form flux by one unit. Having such a $D3$-brane solution requires the presence of five-form flux in the background to stabilize it. Therefore, we cannot put more that N such $D3$-branes as inside the last one there is no more five-form flux left and the $N+1$-th $D3$-brane cannot be stabilized.

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of correlation functions involving Wilson loops. It would also be interesting to consider the case of a circular Wilson loop[5] and study the defect field theory origin of the matrix model proposed by Erickson, Semenoff and Zarembo [13, 14] for the study of circular Wilson loops. We expect that the description of Wilson loops studied in this paper can also be extended to other interesting gauge theories with reduced supersymmetry and different matter content.

The plan of the rest of the paper is as follows. In section 2 we identify the Wilson loop operators in $\mathcal{N}=4$ SYM that preserve half of the supersymmetries and study the $\mathcal{N}=4$ subalgebra preserved by the half-BPS Wilson loops. Section 3 contains the embeddings of the $D5_k$ and $D3_k$ brane in $\mathrm{AdS}_5\times \mathrm{S}^5$ and we show that they preserve the same symmetries as the half-BPS Wilson loop operators. In section 4 we derive the defect conformal field theory produced by the interaction of the bulk $D5_k/D3_k$ branes with the $D3$ branes that generate the $\mathrm{AdS}_5\times \mathrm{S}^5$ background. We also show that a single $D5_k$-brane corresponds to a half-BPS Wilson loop in the k-th antisymmetric product representation of $SU(N)$ while the $D3_k$-brane corresponds to the k-th symmetric product representation. In section 5 we show that a half-BPS Wilson loop in any representation of $SU(N)$ is described in terms of the collection of $D5$ or $D3$ branes explained in the introduction. Some computations have been relegated to the appendices.

2. Wilson loops in $\mathcal{N}=4$ SYM

A Wilson loop operator in $\mathcal{N}=4$ SYM is labeled by a curve C in superspace and by a representation R of the gauge group G. The data that characterizes a Wilson loop, the curve C and the representation R, label the properties of the external particle that is used to probe the theory. The curve C is identified with the worldline of the superparticle propagating in $\mathcal{N}=4$ superspace while the representation R corresponds to the charge carried by the superparticle.

The curve C is parameterized by $(x^\mu(s), y^I(s), \theta^\alpha_A(s))$ and it encodes the coupling of the charged external superparticle to the $\mathcal{N}=4$ SYM multiplet $(A_\mu, \phi^I, \lambda^A_\alpha)$, where μ (α) is a vector(spinor) index of $SO(1,3)$ while I (A) is a vector (spinor) index of the $SO(6)$ R-symmetry group of $\mathcal{N}=4$ SYM. Gauge invariance of the Wilson loop constraints the curve $x^\mu(s)$ to be closed while $(y^I(s), \theta^\alpha_A(s))$ are arbitrary curves.

The other piece of data entering into the definition of a Wilson loop operator is the choice of representation R of the gauge group G. For gauge group $SU(N)$, the irreducible representations are conveniently summarized by a Young tableau $R=(n_1,n_2,\ldots,n_N)$, where n_i is the number of boxes in the i-th row of the tableau and $n_1\geq n_2\geq\ldots\geq n_N\geq 0$.

The corresponding Young diagram is given by:

$$
\begin{array}{|c|c|c|c|c|c|c|}
\hline
1 & 2 & \cdot & \cdot & \cdot & \cdot & n_1 \\
\hline
1 & 2 & \cdot & \cdot & \cdot & n_2 & \multicolumn{1}{c}{} \\
\cline{1-6}
1 & 2 & \cdot & \cdot & \cdot & n_3 & \multicolumn{1}{c}{} \\
\cline{1-6}
\cdot & \cdot & \cdot & \cdot & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\
\cline{1-4}
1 & 2 & \cdot & n_N & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\
\cline{1-4}
\end{array}
$$

[5] Which breaks a different set of supersymmetries compared to the loops considered in this paper.

The main goal of this paper is to identify all half-BPS Wilson loop operators of $\mathcal{N}=4$ SYM in the dual asymptotically AdS gravitational description.

In this paper we consider bosonic Wilson loop operators for which $\theta^{\alpha}_{A}(s)=0$. Wilson loop operators coupling to fermions can be obtained by the action of supersymmetry and are descendant operators. The operators under study are given by

$$W_R(C) = \mathrm{Tr}_R \, P \exp\left(i \int_C ds (A_\mu \dot{x}^\mu + \phi_I \dot{y}^I) \right), \tag{2.1}$$

where C labels the curve $(x^\mu(s), y^I(s))$ and P denotes path-ordering along the curve C.

We now consider the Wilson loop operators in $\mathcal{N}=4$ SYM which are invariant under one-half of the $\mathcal{N}=4$ Poincare supersymmetries and also invariant under one-half of the $\mathcal{N}=4$ superconformal supersymmetries. The sixteen Poincare supersymmetries are generated by a ten dimensional Majorana-Weyl spinor ϵ_1 of negative chirality while the superconformal supersymmetries are generated by a ten dimensional Majorana-Weyl spinor ϵ_2 of positive chirality. The analysis in appendix A shows that supersymmetry restricts the curve C to be a straight time-like line spanned by $x^0 = t$ and $\dot{y}^I = n^I$, where n^I is a unit vector in R^6. The unbroken supersymmetries are generated by $\epsilon_{1,2}$ satisfying

$$\gamma_0 \gamma_I n^I \epsilon_1 = \epsilon_1 \qquad \gamma_0 \gamma_I n^I \epsilon_2 = -\epsilon_2. \tag{2.2}$$

Therefore, the half-BPS Wilson loop operators in $\mathcal{N}=4$ SYM are given by

$$W_R = W_{(n_1,n_2,\ldots,n_N)} = \mathrm{Tr}_R \, P \exp\left(i \int dt \, (A_0 + \phi) \right), \tag{2.3}$$

where $\phi = \phi_I n^I$. It follows that the half-BPS Wilson loop operators carry only one label; the choice of representation R.

We conclude this section by exhibiting the supersymmetry algebra preserved by the insertion of (2.3) to the $\mathcal{N}=4$ path integral. This becomes useful when identifying the gravitational dual description of Wilson loops in later sections. In the absence of any operator insertions, $\mathcal{N}=4$ SYM is invariant under the $PSU(2,2|4)$ symmetry group. It is well known [15] that a straight line breaks the four dimensional conformal group $SU(2,2) \simeq SO(2,4)$ down to $SO(4^*) \simeq SU(1,1) \times SU(2) \simeq SL(2,R) \times SU(2)$. Moreover, the choice of a unit vector n^I in (2.3) breaks the $SU(4) \simeq SO(6)$ R-symmetry of $\mathcal{N}=4$ SYM down to $Sp(4) \simeq SO(5)$. The projections (2.2) impose a reality condition on the four dimensional supersymmetry generators, which now transform in the $(4,4)$ representation of $SO(4^*) \times Sp(4)$. Therefore, the supersymmetry algebra preserved[6].

3. Giant and dual giant Wilson loops

The goal of this section is to put forward plausible candidate D-branes for the bulk description of the half-BPS Wilson loop operators (2.3). In the following sections we show

[6]This supergroup has appeared in the past in relation to the baryon vertex by the half-BPS Wilson loops is $Osp(4^*|4)$

that integrating out the physics on these D-branes results in the insertion of a half-BPS Wilson loop operator to $\mathcal{N}=4$ SYM. This provides the string theory realization of all half-BPS Wilson loops in $\mathcal{N}=4$ SYM.

Given the extended nature of Wilson loop operators in the gauge theory living at the boundary of AdS, it is natural to search for extended objects in $\mathrm{AdS}_5\times \mathrm{S}^5$ preserving the same symmetries as those preserved by the half-BPS operators (2.3) as candidates for the dual description of Wilson loops. The extended objects that couple to the Wilson loop must be such that they span a time-like line in the boundary of AdS, where the Wilson loop operator (2.3) is defined.

Since we want to identify extended objects with Wilson loops in $\mathcal{N}=4$ SYM on $R^{1,3}$, it is convenient to write the AdS_5 metric in Poincare coordinates

$$ds^2_{AdS} = L^2\left(u^2\eta_{\mu\nu}dx^\mu dx^\nu + \frac{du^2}{u^2}\right), \tag{3.1}$$

where $L=(4\pi g_s N)^{1/4} l_s$ is the radius of AdS_5 and S^5. Furthermore, since the Wilson loop operator (2.3) preserves an $SO(5)$ symmetry, we make this symmetry manifest by foliating the metric on S^5 by a family of S^4's

$$ds^2_{sphere} = L^2\left(d\theta^2 + \sin^2\theta\, d\Omega_4^2\right), \tag{3.2}$$

where θ measures the latitude angle of the S^4 from the north pole and $d\Omega_4^2$ is the metric on the unit S^4.

In [1, 2] the bulk description of a Wilson loop in the fundamental representation of the gauge group associated with a curve C in $R^{1,3}$ was given in terms of a fundamental string propagating in the bulk and ending at the boundary of AdS along the curve C. This case corresponds to the simplest Young tableau $R=(1,0,\ldots,0)$, with Young diagram $\square$.

The expectation value of the corresponding Wilson loop operator is identified with the action of the string ending at the boundary along C. This identification was motivated by considering a stack of D3-branes and moving one of them to infinity, leaving behind a massive external particle carrying charge in the fundamental representation of the gauge group.

The embedding corresponding to the half-BPS Wilson loop (2.3) for $R=(1,0,\ldots,0)$ is given by[7]

$$\sigma^0 = x^0 \qquad \sigma^1 = u \qquad x^i = 0 \qquad x^I = n^I, \tag{3.3}$$

so that the fundamental string spans an AdS_2 geometry sitting at $x^i=0$ in AdS_5 and sits at a point on the S^5 labeled by a unit vector n^I, satisfying $n^2=1$. Therefore, the fundamental string preserves exactly the same $SU(1,1)\times SU(2)\times SO(5)$ symmetries as the one-half BPS Wilson loop operator (2.3). Moreover the string ends on the time-like line parameretrized by $x^0=t$, which is the curve corresponding to the half-BPS Wilson loop (2.3).

[7]The coordinates $\sigma^0,\ldots\sigma^p$ refer to the worldvolume coordinates on a string/brane.

In appendix B we compute the supersymmetries left unbroken by the fundamental string (3.3). We find that they are generated by two ten dimensional Majorana-Weyl spinors $\epsilon_{1,2}$ of opposite chirality satisfying

$$\gamma_0\gamma_I n^I \epsilon_1 = \epsilon_1 \qquad \gamma_0\gamma_I n^I \epsilon_2 = -\epsilon_2, \tag{3.4}$$

which coincides with the unbroken supersymmetries (2.2) of the half-BPS Wilson loop. Therefore, the fundamental string preserves the same $Osp(4^*|4)$ symmetry as the half-BPS Wilson loop (2.3).

The main question in this paper is, what is the holographic description of half-BPS Wilson loop operators in higher representations of the gauge group?

Intuitively, higher representations correspond to having multiple coincident fundamental strings[8] ending at the boundary of AdS. This description is, however, not very useful as the Nambu-Goto action only describes a single string. A better description of the system is achieved by realizing that coincident fundamental strings in the $\mathrm{AdS}_5\times\mathrm{S}^5$ background can polarize [19] into a single D-brane with fundamental strings dissolved in it, thus providing a concrete description of the coincident fundamental strings.

We now describe the way in which a collection of k fundamental strings puff up into a D-brane with k units of fundamental string charge on the D-brane worldvolume.

The guide we use to determine which D-branes are the puffed up description of k-fundamental strings is to consider D-branes in $\mathrm{AdS}_5\times\mathrm{S}^5$ which are invariant under the same symmetries as the half-BPS Wilson loops[9], namely we demand invariance under $Osp(4^*|4)$. The branes preserving the $SU(1,1)\times SU(2)\times SO(5)$ symmetries of the Wilson loop are given by:
1) $D5_k$-brane with $\mathrm{AdS}_2\times\mathrm{S}^4$ worldvolume.
2) $D3_k$-brane with $\mathrm{AdS}_2\times\mathrm{S}^2$ worldvolume.

We now describe the basic properties of these branes that we need for the analysis in upcoming sections.

3.1 $D5_k$-brane as a Giant Wilson loop

The classical equations of motion for a $D5$-brane with an $\mathrm{AdS}_2\times\mathrm{S}^4$ geometry and with k fundamental strings dissolved in it (which we label by $D5_k$) has been studied in the past in [20, 21]. Here we summarize the necessary elements that will allow us to prove in the following section that this D-brane corresponds to a half-BPS Wilson loop operator.

The $D5_k$-brane is described by the following embedding

$$\sigma^0 = x^0 \qquad \sigma^1 = u \qquad \sigma^a = \varphi_a \qquad x^i = 0 \qquad \theta = \theta_k = \text{constant}, \tag{3.5}$$

together with a nontrivial electric field F along the AdS_2 spanned by (x^0, u). Therefore, a $D5_k$-brane spans an $\mathrm{AdS}_2\times\mathrm{S}^4$ geometry[10] and sits at a latitude angle $\theta = \theta_k$ on the S^5, which depends on k, the fundamental string charge carried by the $D5_k$-brane:

[8]Such a proposal was put forward in [18] by drawing lessons from the description of Wilson loops in two dimensonal QCD.

[9]We have already established that the fundamental strings (3.3) have the same symmetries as the half-BPS Wilson loops.

[10]φ_a are the coordinates on the S^4 in (3.2).

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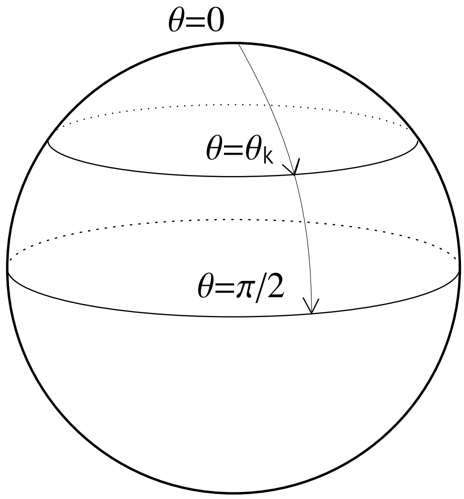

Figure 2: A $D5_k$-brane sits at a latitude angle θ_k determined by the amount of fundamental string charge it carries.

This brane describes the puffing up of k fundamental strings into a D-brane inside S^5, so in analogy with a similar phenomenon for point-like gravitons [22], such a brane can be called a giant Wilson loop.

It can be shown [21] that θ_k is a monotonically increasing function of k in the domain of θ, that is $[0, \pi]$ and that $\theta_0 = 0$ and $\theta_N = \pi$, where N is the amount of flux in the $AdS_5 \times S^5$ background or equivalently the rank of the gauge group in $\mathcal{N} = 4$ SYM. Therefore, we can dissolve at most N fundamental strings on the $D5$-brane.

The $D5_k$-brane has the same bosonic symmetries as the half-BPS Wilson loop operator and it ends on the boundary of AdS_5 along the time-like line where the half-BPS Wilson loop operator (2.3) is defined. In appendix B we show that it also preserves the same supersymmetries (2.2) as the half-BPS Wilson loop operator (2.3) when $n^I = (1, 0, \ldots, 0)$ and is therefore invariant under the $Osp(4^*|4)$ symmetry group.

3.2 $D3_k$-brane as a Dual Giant Wilson loop

The classical equations of motion of a $D3$-brane with an $AdS_2 \times S^2$ geometry and with k fundamental strings dissolved in it (which we label by $D3_k$) has been studied recently by Drukker and Fiol [7]. We refer the reader to this reference for the details of the solution.

For our purposes we note that unlike for the case of the $D5_k$-brane, an arbitrary amount of fundamental string charge can be dissolved on the $D3_k$-brane. As we shall see in the next section, this has a pleasing interpretation in $\mathcal{N} = 4$.

The geometry spanned by a $D3_k$-brane gives an $AdS_2 \times S^2$ foliation[11] of AdS_5, the location of the slice being determined by k, the amount of fundamental string charge:

[11] This foliation structure and the relation with $\mathcal{N} = 4$ SYM defined on the $AdS_2 \times S^2$ boundary — which makes manifest the symmetries left unbroken by the insertion of a straight line Wilson loop — has been considered in [23, 24, 12]

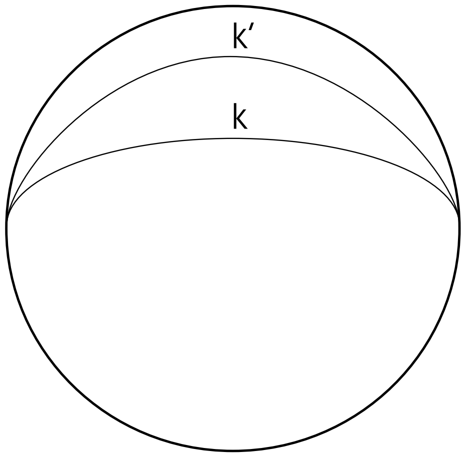

Figure 3: A $D3_k$-brane gives an $\mathrm{AdS}_2 \times \mathrm{S}^2$ slicing of AdS_5.

This brane describes the puffing up of k fundamental strings into a D-brane inside AdS_5, so in analogy with a similar phenomenon for point-like gravitons [25, 26], such a brane can be called a dual giant Wilson loop.

By generalizing the supersymmetry analysis in [7] one can show that the $D3_k$-brane preserves precisely the same supersymmetries as the fundamental string (3.4) and therefore the same as the ones preserved by the half-BPS Wilson loop operator.

To summarize, we have seen that k fundamental strings can be described either by a single $D5_k$-brane or by a single $D3_k$-brane. The three objects preserve the same $Osp(4^*|4)$ symmetry if the fundamental string and the $D3_k$-brane sit at the north pole of the S^5, i.e. at $\theta = 0$ corresponding to the unit vector $n^I = (1, 0, \ldots, 0)$. Furthermore, these three objects are invariant under the same $Osp(4^*|4)$ symmetry as the half-BPS Wilson loop operator (2.3).

4. Dirichlet Branes as Wilson loops

We show that the half-BPS Wilson loop operators in $\mathcal{N} = 4$ SYM are realized by the D-branes in the previous section. We study the modification on the low energy effective field theory on the N $D3$-branes that generate the $\mathrm{AdS}_5 \times \mathrm{S}^5$ background due to the presence of $D5$-brane giants and $D3$-brane dual giants. We can integrate out exactly the degrees of freedom introduced by the Wilson loop D-branes and show that the net effect of these D-branes is to insert into the $\mathcal{N} = 4$ $SU(N)$ SYM path integral a Wilson loop operator in the desired representation of the $SU(N)$ gauge group.

In order to develop some intuition for how this procedure works, we start by analyzing the case of a single $D5_k$-brane and a single $D3_k$-brane. We now show that a $D5_k$-brane describes a half-BPS Wilson loop operator in the k-th antisymmetric product representation of $SU(N)$ while a $D3_k$-brane describes one in the k-th symmetric product representation.

In section 5 we proceed to show that a Wilson loop described by an arbitrary Young tableau corresponds to considering multiple D-branes. We also show that a given Young tableau can be either derived from a collection of $D5_k$-branes or from a collection of $D3_k$-branes and that the two descriptions are related by bosonization.

4.1 $D5_k$-brane as a Wilson Loop

We propose to analyze the physical interpretation of a single $D5_k$-brane in the gauge theory by studying the effect it has on four dimensional $\mathcal{N} = 4$ SYM. A $D5_k$-brane with an $\text{AdS}_2\times\text{S}^4$ worldvolume in $\text{AdS}_5\times\text{S}^5$ arises in the near horizon limit of a single $D5$-brane probing the N $D3$-branes that generate the $\text{AdS}_5\times\text{S}^5$ background. The flat space brane configuration is given by:

$$\begin{array}{ccccccccccc} & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ D3 & \text{X} & \text{X} & \text{X} & \text{X} & & & & & & \\ D5 & \text{X} & & & & & \text{X} & \text{X} & \text{X} & \text{X} & \text{X} \end{array} \tag{4.1}$$

We can now study the effect of the $D5_k$-brane by analyzing the low energy effective field theory on a single $D5$-brane probing N $D3$-branes in flat space

We note first that the $D5$-brane produces a codimension three defect on the $D3$-branes, since they overlap only in the time direction. In order to derive the decoupled field theory we must analyze the various open string sectors. The 3-3 strings give rise to the the familiar four dimensional $\mathcal{N} = 4$ $SU(N)$ SYM theory. The sector of 3-5 and 5-3 strings give rise to degrees of freedom that are localized in the defect. There are also the 5-5 strings. The degrees of freedom associated with these strings — a six dimensional vector multiplet on the $D5$-brane — are not dynamical. Nevertheless, as we will see, they play a crucial role in encoding the choice of Young tableau $R = (n_1, \ldots, n_N)$.

This brane configuration gives rise to a defect conformal field theory (see e.g. [8, 9]), which describes the coupling of the $\mathcal{N} = 4$ SYM to the localized degrees of freedom. The localized degrees of freedom arise from the 3-5 and 5-3 strings and they give rise to fermionic fields χ transforming in the fundamental representation of $SU(N)$. We can write the action of this defect conformal field theory by realizing that we can obtain it by performing T-duality on the well studied D0-D8 matrix quantum mechanics (see e.g. [27, 28]). Ignoring for the moment the coupling of χ to the non-dynamical 5-5 strings, we obtain that the action of our defect conformal field theory is given by[12]

$$S = S_{\mathcal{N}=4} + \int dt\; i\chi^\dagger\partial_t\chi + \chi^\dagger(A_0 + \phi)\chi, \tag{4.2}$$

where A_0 is the temporal component of the gauge field in $\mathcal{N} = 4$ SYM and ϕ is one of the scalars of $\mathcal{N} = 4$ SYM describing the position of the $D3$-branes in the direction transverse to both the $D3$ and $D5$ branes; it corresponds to the unit vector $n^I = (1, 0, \ldots, 0)$.

What are the $PSU(2,2|4)$ symmetries that are left unbroken by adding to the $\mathcal{N} = 4$ action the localized fields? The supersymmetries of $\mathcal{N} = 4$ SYM act trivially on χ. This implies that the computation determining the unbroken supersymmetries is exactly the

[12] We do not write the $SU(N)$ indices explicitly. They are contracted in a straighforward manner between the χ_i fields and the $A_{0\,ij}$ gauge field, where $i, j = 1, \ldots, N$.

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same as the one we did for the Wilson loop operator (2.3). Likewise for the bosonic symmetries, where we just need to note that the defect fields live on a time-like straight line. Therefore, we conclude that our defect conformal field theory has an $Osp(4^*|4)$ symmetry, just like the half-BPS Wilson loop operator (2.3).

Even though the fields arising from the 5-5 strings are nondynamical, they play a crucial role in the identification of the $D5_k$-brane with a Wilson loop operator in a particular representation of the gauge group. As we discussed in the previous section, a $D5_k$-brane has k fundamental strings ending on it and we must find a way to encode the choice of k in the low energy effective field theory on the D-branes in flat space. This can be accomplished by recalling that a fundamental string ending on a D-brane behaves as an electric charge for the gauge field living on the D-brane. Therefore we must add to (4.2) a term that captures the fact that there are k units of background electric charge localized on the defect. This is accomplished by inserting into our defect conformal field theory path integral the operator:

$$\exp\left(-ik\int dt\ \tilde{A}_0\right). \tag{4.3}$$

Equivalently, we must add to the action (4.2) the Chern-Simons term:

$$-\int dt\ k\tilde{A}_0. \tag{4.4}$$

The effect of (4.4) on the $\tilde{A}_0$ equation of motion is to insert k units of electric charge at the location of the defect, just as desired.

We must also consider the coupling of the χ fields to the nondynamical gauge field $\tilde{A}$ on the $D5$-brane, as they transform in the fundamental representation of the $D5$-brane gauge field. Summarizing, we must add to (4.2) :

$$S_{extra} = \int dt\ \chi^\dagger \tilde{A}_0 \chi - k\tilde{A}_0. \tag{4.5}$$

The addition of these extra couplings preserves the $Osp(4^*|4)$ symmetry of our defect conformal field theory.

We want to prove that a $D5_k$-brane corresponds to a half-BPS Wilson loop operator in $\mathcal{N}=4$ SYM in a very specific representation of $SU(N)$. The way we show this is by integrating out explicitly the degrees of freedom associated with the $D5_k$-brane. We must calculate the following path integral

$$Z = \int [D\chi][D\chi^\dagger][D\tilde{A}_0]\ e^{i(S+S_{extra})}, \tag{4.6}$$

where S is given in (4.2) and S_{extra} in (4.5).

Let's us ignore the effect of S_{extra} for the time being; we will take it into account later. We first integrate out the χ fields. This can be accomplished the easiest by perfoming a choice of gauge such that the matrix $A_0 + \phi$ has constant eigenvalues[13]:

$$A_0 + \phi = \mathrm{diag}(w_1, \ldots, w_N). \tag{4.7}$$

The equations of motion for the χ fields are then given by:

$$(i\partial_t + w_i)\chi_i = 0 \qquad \text{for} \qquad i = 1, \ldots, N. \tag{4.8}$$

Therefore, in this gauge, one has a system of N fermions χ_i with energy w_i.

The path integral can now be conveniently evaluated by going to the Hamiltonian formulation, where integrating out the χ fermions corresponds to evaluating the partition function of the fermions[14]. Therefore, we are left with

$$Z^* = e^{iS_{\mathcal{N}=4}} \cdot \prod_{i=1}^{N}(1 + x_i), \tag{4.9}$$

where $x_i = e^{i\beta w_i}$ and the $*$ in (4.9) is to remind us that we have not yet taken into account the effect of S_{extra} in (4.6). A first glimpse of the connection between a $D5_k$-brane and a half-BPS Wilson loop operator is to recognize that the quantity $x_i = e^{i\beta w_i}$ appearing in (4.9) with w_i given in (4.7), is an eigenvalue of the holonomy matrix appearing in the Wilson loop operator (2.3), that is $\exp i\beta\,(A_0 + \phi)$.

Since our original path integral (4.6) is invariant under $SU(N)$ conjugations, it means that Z^* should have an expansion in terms of characters or invariant traces of $SU(N)$, which are labeled by a Young tableau $R = (n_1, n_2, \ldots, n_N)$. In order to exhibit which representations R appear in the partition function, we split the computation of the partition function into sectors with a fixed number of fermions in a state. This decomposition allows us to write

$$\prod_{i=1}^{N}(1 + x_i) = \sum_{l=0}^{N} E_l(x_1, \ldots, x_l), \tag{4.10}$$

where $E_l(x_1, \ldots, x_l)$ is the symmetric polynomial:

$$E_l(x_1, \ldots, x_l) = \sum_{i_1 < i_2 \ldots < i_l} x_{i_1} \ldots x_{i_l}. \tag{4.11}$$

Physically, $E_l(x_1, \ldots, x_l)$ is the partition function over the Fock space of N fermions, each with energy w_i, that have l fermions in a state.

We now recognize that the polynomial E_l is the formula (see e.g. [29]) for the trace of the half-BPS Wilson loop holonomy matrix in the l-th antisymmetric representation

$$\begin{aligned} E_l &= \mathrm{Tr}_{(\underbrace{1, \ldots, 1}, 0, \ldots, 0)} \\ P\exp\left(i\int dt\;(A_0 + \phi)\right) &= W_{(\underbrace{1, \ldots, 1}, 0, \ldots, 0)}\,, \end{aligned} \tag{4.12}$$

[13]Here there is a subtlety. This gauge choice introduces a Fadeev-Popov determinant which changes the measure of the path-integral over the $\mathcal{N} = 4$ SYM fields. Nevertheless, after we integrate out the degrees of freedom associated with the $D5$-brane, we can write the result in a gauge invariant form, so that the Fadeev-Popov determinant can be reabsorbed to yield the usual measure over the $\mathcal{N} = 4$ SYM fields in the path integral.

[14]Here we introduce, for convenience an infrared regulator, so that t is compact $0 \leq t \leq \beta$.

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where $W_{(\underbrace{1,\dots,1}_{l},0,\dots,0)}$ is the half-BPS Wilson loop operator (2.3) corresponding to the following Young diagram:

$$\begin{array}{|c|}\hline 1\\ \hline 2\\ \hline \cdot\\ \hline \cdot\\ \hline \cdot\\ \hline l\\ \hline\end{array}$$

Therefore, integrating out the χ fields has the effect of inserting into the $\mathcal{N}=4$ path integral a sum over all half-BPS Wilson loops in the l-th antisymmetric representation:

$$Z^* = e^{iS_{\mathcal{N}=4}} \cdot \sum_{l=0}^{N} W_{(\underbrace{1,\dots,1},0,\dots,0)} \cdot \tag{4.13}$$

It is now easy to go back and consider the effect of S_{extra} (4.5) on the path integral (4.6). Integrating over $\tilde{A}_0$ in (4.6) imposes the following constraint:

$$\sum_{i=1}^{N} \chi_i^\dagger \chi_i = k. \tag{4.14}$$

This constraint restrict the sum over states in the partition function to states with precisely k fermionic excitations. These states are of the form:

$$\chi_{i_1}^\dagger \dots \chi_{i_k}^\dagger |0\rangle. \tag{4.15}$$

This picks out the term with $l=k$ in (4.13) .

Therefore, we have shown that a single $D5_k$-brane inserts a half-BPS operator in the k-th antisymmetric representation in the $\mathcal{N}=4$ path integral

$$D5_k \longleftrightarrow Z = e^{iS_{\mathcal{N}=4}} \cdot W_{(\underbrace{1,\dots,1},0,\dots,0)}\,, \tag{4.16}$$

where $S_{\mathcal{N}=4}$ is the action of $\mathcal{N}=4$ SYM. The expectation value of this operator can be computed by evaluating the classical action of the $D5_k$-brane.

4.2 $D3_k$-brane as a Wilson Loop

We now consider what a $D3_k$ dual giant brane corresponds to in four dimensional $\mathcal{N}=4$ SYM. Here we run into a puzzle. Unlike for the case of a $D5_k$-brane, where we could study the physics produced by the brane by identifying a brane configuration in flat space that gives rise to a $D5_k$-brane in $\mathrm{AdS}_5\times\mathrm{S}^5$ in the near horizon/decoupling limit, there is no brane configuration in flat space that gives rise in the near horizon/decoupling limit to a $D3_k$-brane in $\mathrm{AdS}_5\times\mathrm{S}^5$.

Despite this shortcoming we make a very simple proposal for how to study the effect produced by a $D3_k$-brane on $\mathcal{N}=4$ SYM and show that it leads to a consistent physical picture. The basic observation is that if we quantize the χ fields appearing in (4.2) (4.5) not

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as fermions but as bosons, which is something that is consistent when quantizing degrees of freedom in $0+1$ dimensions, we can show that the effect of the $D3_k$-brane is to insert a half-BPS Wilson loop operator (2.3) in the k-th symmetric representation of $SU(N)$.

This result is in concordance with the basic physics of the probe branes. In the previous section we found that the amount of fundamental string charge k on a $D5_k$-brane can be at most N. On the other hand, we have shown that a $D5_k$-brane corresponds to a Wilson loop in the k-th antisymmetric representation of $SU(N)$ so that indeed $k \leq N$, otherwise the operator vanishes. For the $D3_k$-brane, however, the string charge k can be made arbitrarily large. The proposal that the $D3_k$-brane can be studied in the gauge theory by quantizing χ as bosons leads, as we will show, to a Wilson loop in the k-th symmetric representation, for which there is a non-trivial representation of $SU(N)$ for all k and fits nicely with the $D3_k$-brane probe expectations.

Formally, going from the $D5_k$ giant to the $D3_k$ dual giant Wilson line picture amounts to performing a bosonization of the defect field χ . It would be very interesting to understand from a more microscopic perspective the origin of this bosonization[15].

Having motivated treating χ as a boson we can now go ahead and integrate out the χ fields in (4.6). As before, we ignore for the time being the effect of S_{extra} in (4.6). We also diagonalize the matrix $A_0 + \phi$ as in (4.7).

The equations of motion are now those for N chiral bosons χ_i with energy w_i

$$(i\partial_t + w_i)\chi_i = 0 \qquad \text{for} \qquad i = 1, \ldots, N, \tag{4.17}$$

where w_i are the eigenvalues of the matrix $A_0 + \phi$.

The path integral over χ in (4.6) is computed by evaluating the partition function of the chiral bosons, which yield

$$Z^* = e^{iS_{\mathcal{N}=4}} \cdot \prod_{i=1}^{N} \frac{1}{1-x_i}, \tag{4.18}$$

where $x_i = e^{i\beta w_i}$ and the $*$ in (4.9) is to remind us that we have not yet taken into account the effect of S_{extra} in (4.6). x_i are the eigenvalues of the holonomy matrix appearing in the Wilson loop operator (2.3).

In order to connect this computation with Wilson loops in $\mathcal{N} = 4$ SYM it is convenient to decompose the Fock space of the chiral bosons in terms of subspaces with a fixed number of bosons in a state. This decomposition yields

$$\prod_{i=1}^{N} \frac{1}{1-x_i} = \sum_{l=0}^{\infty} H_l(x_1, \ldots, x_l), \tag{4.19}$$

where $H_l(x_1, \ldots, x_l)$ is the symmetric polynomial:

$$H_l(x_1, \ldots, x_l) = \sum_{i_1 \leq i_2 \ldots \leq i_l} x_{i_1} \ldots x_{i_l}. \tag{4.20}$$

[15] A similar type of bosonization seems to be at play in the description of half-BPS local operators in $\mathcal{N} = 4$ SYM in terms of giants and dual giant gravitons [30].

Physically, $H_l(x_1,\ldots,x_l)$ is the partition function over the Fock space of N chiral bosons with energy w_i that have l bosons in a state.

We now recognize that the polynomial H_l is the formula (see e.g. [29]) for the trace of the half-BPS Wilson loop holonomy matrix in the l-th symmetric representation

$$H_l = \mathrm{Tr}_{(l,0,\ldots,0)}\, P\exp\left(i\int dt\,(A_0+\phi)\right) = W_{(l,0,\ldots,0)}, \tag{4.21}$$

where $W_{(l,0,\ldots,0)}$ is the half-BPS Wilson loop operator (2.3) corresponding to the following Young diagram:

$$\boxed{1}\boxed{2}\boxed{\cdot}\boxed{\cdot}\boxed{\cdot}\boxed{\cdot}\boxed{l}$$

Therefore, integrating out the χ fields has the effect of inserting into the $\mathcal{N}=4$ path integral a sum over all half-BPS Wilson loops in the l-th symmetric representation:

$$Z^* = e^{iS_{\mathcal{N}=4}}\cdot\sum_{l=0}^{N} W_{(l,0,\ldots,0)}. \tag{4.22}$$

It is now straightforward to take into account the effect of S_{extra} (4.5) in (4.6). Integrating over $\tilde{A}_0$ imposes the constraint (4.14). This constraint picks out states with precisely k bosons (4.15) and therefore selects the term with $l=k$ in (4.19).

Therefore, we have shown that a single $D3_k$-brane inserts a half-BPS operator in the k-th symmetric representation in the $\mathcal{N}=4$ path integral

$$D3_k \longleftrightarrow Z = e^{iS_{\mathcal{N}=4}}\cdot W_{(k,0,\ldots,0)}, \tag{4.23}$$

where $S_{\mathcal{N}=4}$ is the action of $\mathcal{N}=4$ SYM. The expectation value of this operator can be computed by evaluating the classical action of the $D3_k$-brane.

5. D-brane description of an Arbitrary Wilson loop

In the previous section we have shown that Wilson loops labeled by Young tableaus with a single column are described by a $D5$-brane while a $D3$-brane gives rise to tableaus with a single row. What is the gravitational description of Wilson loops in an arbitrary representation?

We now show that given a Wilson loop operator described by an arbitrary Young tableau, that it can be described either in terms of a collection of giants or alternatively in terms of a collection of dual giants.

5.1 Wilson loops as $D5$-branes

In the previous section, we showed that the information about the number of boxes in the Young tableau with one column is determined by the amount of fundamental string charge ending on the $D5$-brane. For the case of a single $D5_k$-brane, this background electric charge is captured by inserting (4.3)

$$\exp\left(-ik\int dt\,\tilde{A}_0\right) \tag{5.1}$$

in the path integral of the defect conformal field theory. Equivalently, we can add the Chern-Simons term:

$$-\int dt\, k\tilde{A}_0 \tag{5.2}$$

to the action (4.2). This injects into the theory a localized external particle of charge k with respect to the $U(1)$ gauge field $\tilde{A}_0$ on the $D5$-brane.

We now show that describing half-BPS Wilson loop operators (2.3) labeled by tableaus with more that one column corresponds to considering the brane configuration in (4.1) with multiple $D5$-branes.

In order to show this, we must consider the low energy effective field theory on M $D5$-branes probing N $D3$-branes. In this case, the $U(1)$ symmetry associated with the $D5$-brane gets now promoted to a $U(M)$ symmetry, where M is the number of $D5$-branes. Therefore, the defect conformal field theory living on this brane configuration is given by[16]

$$S = S_{\mathcal{N}=4} + \int dt\, i\chi_i^{I\dagger}\partial_t\chi_i^I + \chi_i^{I\dagger}(A_{0ij}+\phi_{ij})\chi_j^I, \tag{5.3}$$

where i,j is a fundamental index of $SU(N)$ while I,J is a fundamental index of $U(M)$.

We need to understand how to realize in our defect conformal field theory that we have M $D5$-branes in $\mathrm{AdS}_5\times\mathrm{S}^5$ with a configuration of fundamental strings dissolved in them. Physically, the string endpoints introduce into the system a background charge for the $U(M)$ gauge field which depends on the distribution of string charge among the M $D5$-branes. The charge is labeled by a representation $\rho = (k_1,\ldots,k_M)$ of $U(M)$, where now $\rho = (k_1,\ldots,k_M)$ is a Young tableau of $U(M)$. A charge in the representation $\rho = (k_1,\ldots,k_M)$ is produced when k_i fundamental strings end on the i-th $D5$-brane. This $D5$-brane configuration can be labeled by the array $(D5_{k_1},\ldots,D5_{k_M})$:

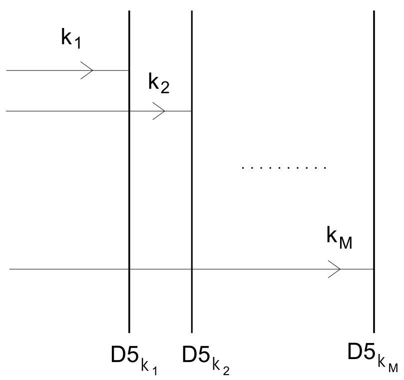

Figure 4: Array of strings producing a background charge given by the representation $\rho = (k_1,\ldots,k_M)$ of $U(M)$. The $D5$-branes are drawn separated for illustration purposes only, as they sit on top of each other.

[16]For clarity, we write explicitly the indices associated with $SU(N)$ and $U(M)$.

We must now add to the defect conformal field theory a term that captures that there is a static background charge $\rho = (k_1, \ldots, k_M)$ induced in the system by the fundamental strings. This is accomplished by inserting into the path integral a Wilson loop operator for the gauge field $\tilde{A}_0$. The operator insertion is given by

$$\text{Tr}_{(k_1,k_2,\ldots,k_M)}\, P\exp\left(-i\int dt\; \tilde{A}_0\right), \tag{5.4}$$

which generalizes (5.1) when there are multiple $D5$-branes. We must also take into account the coupling of the localized fermions χ_i^I to $\tilde{A}_0$:

$$S_{extra} = \int dt\; \chi_i^{I\dagger}\tilde{A}_{0\,IJ}\chi_i^J. \tag{5.5}$$

In order to study what the $(D5_{k_1}, \ldots, D5_{k_M})$ array in $\text{AdS}_5\times\text{S}^5$ corresponds to in $\mathcal{N}=4$ SYM, we need to calculate the following path integral

$$Z = \int [D\chi][D\chi^\dagger][D\tilde{A}_0]\; e^{i(S+S_{extra})}\cdot \text{Tr}_{(k_1,k_2,\ldots,k_M)}\, P\exp\left(-i\int dt\; \tilde{A}_0\right), \tag{5.6}$$

where S is given in (5.3) and S_{extra} in (5.5).

We proceed by gauge fixing the $SU(N)\times U(M)$ symmetry of the theory by diagonalizing $A_0+\phi$ and $\tilde{A}_0$ to have constant eigenvalues respectively. The eigenvalues are given by:

$$\begin{aligned} A_0+\phi &= \text{diag}(w_1, \ldots, w_N) \\ \tilde{A}_0 &= \text{diag}(\Omega_1, \ldots, \Omega_M). \end{aligned} \tag{5.7}$$

Since the path integral in (5.6) involves integration over $\tilde{A}_0$ care must be taken in doing the gauge fixing procedure[17]. As shown in appendix C, the measure over the Hermitean matrix $\tilde{A}_0$ combines with the Fadeev-Popov determinant Δ_{FP} associated with the gauge choice

$$\tilde{A}_0 = \text{diag}(\Omega_1, \ldots, \Omega_M) \tag{5.8}$$

to yield the measure over a unitary matrix U. That is

$$[D\tilde{A}_0]\cdot\Delta_{FP} = [DU], \tag{5.9}$$

with $U = e^{i\beta\tilde{A}_0}$ and

$$[DU] = \prod_{I=1}^{M} d\Omega_I\; \Delta(\Omega)\bar{\Delta}(\Omega), \tag{5.10}$$

where $\Delta(\Omega)$ is the Vandermonde determinant:[18]

$$\Delta(\Omega) = \prod_{I<J}(e^{i\beta\Omega_I} - e^{i\beta\Omega_J}). \tag{5.11}$$

[17] As discussed in footnote 11, the gauge fixing associated with the $SU(N)$ symmetry can be undone once one is done integrating out over χ and $\tilde{A}_0$.

[18] There is a residual $U(1)^N$ gauge symmetry left over after the gauge fixing (5.8) which turns Ω_I into angular coordinates. We are then left with the proper integration domain over the angles of a unitary matrix.

In this gauge, another simplification occurs. The part of the action in (5.6) depending on the χ fields is given by:

$$\int dt\ \chi_i^{I\dagger}(\partial_t + w_i + \Omega_I)\chi_i^I. \tag{5.12}$$

Correspondingly, the equations of motion are:

$$(i\partial_t + w_i + \Omega_I)\chi_i^I = 0 \qquad \text{for} \qquad i = 1, \ldots, N \quad I = 1, \ldots, M. \tag{5.13}$$

Therefore, we have a system of $N \cdot M$ fermions χ_i^I with energy $w_i + \Omega_I$.

We can explicitly integrate out the χ fields in Z (5.6) by going to the Hamiltonian formulation, just as before. The fermion partition function is:

$$\prod_{i=1}^{N} \prod_{J=1}^{M} (1 + x_i e^{i\beta\Omega_J}), \tag{5.14}$$

where as before $x_i = e^{i\beta w_i}$ is an eigenvalue of the holonomy matrix appearing in the Wilson loop operator (2.3) and $e^{i\beta\Omega_J}$ is an eigenvalue of the unitary matrix U.

Combining this with the computation of the measure, the path integral (5.6) can be written as

$$Z = e^{iS_{\mathcal{N}=4}} \cdot \int [DU]\ \chi_{(k_1,\ldots,k_M)}(U^*) \prod_{i=1}^{N} \prod_{J=1}^{M} (1 + x_i e^{i\beta\Omega_J}), \tag{5.15}$$

where we have identified the operator insertion (5.4) with a character in the $\rho = (k_1, \ldots, k_M)$ representation of $U(M)$:

$$\chi_{(k_1,\ldots,k_M)}(U^*) \equiv \mathrm{Tr}_{(k_1,\ldots,k_M)} e^{-i\beta\tilde{A}_0}. \tag{5.16}$$

The partition function of the fermions (5.14) can be expanded either in terms of characters of $SU(N)$ or $U(M)$ by using a generalization of the formula we used in (4.10). We find it convenient to write it in terms of characters of $U(M)$

$$\begin{aligned} \prod_{J=1}^{M} (1 + x_i e^{i\Omega_J}) &= \sum_{l=0}^{M} x_i^l\, \chi_{(\underbrace{1, \ldots, 1}, 0, \ldots, 0)} \\ (U) &= \sum_{l=0}^{M} x_i^l E_l(U_1, \ldots, U_M), \end{aligned} \tag{5.17}$$

where

$$E_l(U) = \mathrm{Tr}_{(\underbrace{1, \ldots, 1}_{l}, 0, \ldots, 0)}\, e^{i\beta\tilde{A}_0} \tag{5.18}$$

is the character of $U(M)$ in the l-th antisymmetric product representation. We recall that $U = e^{i\beta\tilde{A}_0}$ and that $U_I = e^{i\beta\Omega_I}$ for $I = 1, \ldots, M$ are its eigenvalues.

We now use the following mathematical identity

$$\prod_{i=1}^{N} \sum_{l=0}^{M} x_i^l E_l(U) = \sum_{M \geq n_1 \geq n_2 \geq \ldots \geq n_N} \det(E_{n_j+i-j}(U))\ \chi_{(n_1,\ldots,n_N)}(x), \tag{5.19}$$

where

$$\chi_{(n_1,\ldots,n_N)}(x) = W_{(n_1,\ldots,n_N)} \tag{5.20}$$

is precisely the half-BPS Wilson loop operator (2.3) in the $R = (n_1,\ldots,n_N)$ representation of $SU(N)$. Therefore, the fermion partition function (5.14) can be written in terms of $SU(N)$ and $U(M)$ characters as follows

$$\prod_{i=1}^{N}\prod_{J=1}^{M}(1+x_i e^{i\beta\Omega_J}) = \sum_{M\geq n_1\geq n_2\geq\ldots\geq n_N} \det(E_{n_j+i-j}(U))W_{(n_1,\ldots,n_N)}. \tag{5.21}$$

The determinant $\det(E_{n_j+i-j}(U))$ can be explicitly evaluated by using Giambelli's formula (see e.g. [29])

$$\det(E_{n_j+i-j}(U)) = \chi_{(m_1,m_2,\ldots,m_M)}(U), \tag{5.22}$$

where $\chi_{(m_1,m_2,\ldots,m_M)}(U)$ is the character of $U(M)$ associated with the Young tableau $(m_1, m_2, \ldots, m_M)$. This tableau is obtained from $(n_1, n_2, \ldots, n_N)$ by conjugation, which corresponds to transposing the tableau $(n_1, n_2, \ldots, n_N)$:

Figure 5: A Young tableau and its conjugate. In the conjugate tableau the number of boxes in the i-th row is the number of boxes in the i-th column of the original one.

The number of rows in the conjugated tableau $(m_1, m_2, \ldots, m_M)$ is constrained to be at most M due to the $M \geq n_1 \geq n_2 \geq \ldots \geq n_N$ constraint in the sum (5.21).

These computations allow us to write (5.15) in the following way:

$$Z = e^{iS_{\mathcal{N}=4}} \cdot \sum_{M\geq n_1\geq n_2\geq\ldots\geq n_N} W_{(n_1,\ldots,n_N)} \cdot \int [DU]\; \chi_{(m_1,m_2,\ldots,m_M)}(U)\chi_{(k_1,\ldots,k_M)}(U^*). \tag{5.23}$$

Now using orthogonality of $U(M)$ characters:

$$\int [DU]\; \chi_{(m_1,m_2,\ldots,m_M)}(U)\chi_{(k_1,\ldots,k_M)}(U^*) = \prod_{I=1}^{M}\delta_{m_I,k_I}, \tag{5.24}$$

we arrive at the final result

$$Z = e^{iS_{\mathcal{N}=4}} \cdot W_{(l_1,\ldots,l_N)}, \tag{5.25}$$

where $(l_1,\ldots,l_N)$ is the tableau conjugate to $(k_1,\ldots,k_M)$.

To summarize, we have shown that a collection of $D5$-branes described by the array $(D5_{k_1},\ldots,D5_{k_M})$ in $\mathrm{AdS}_5\times\mathrm{S}^5$ corresponds to the half-BPS Wilson loop operator (2.3) in $\mathcal{N}=4$ SYM in the representation $R = (l_1,\ldots,l_N)$ of $SU(N)$

$$(D5_{k_1},\ldots,D5_{k_M}) \longleftrightarrow Z = e^{iS_{\mathcal{N}=4}} \cdot W_{(l_1,\ldots,l_N)}, \tag{5.26}$$

where $(l_1, \ldots, l_N)$ is the tableau conjugate to $(k_1, \ldots, k_M)$. Thererefore, any half-BPS Wilson loop operator in $\mathcal{N} = 4$ has a bulk realization. We now move on to show that there is an alternative bulk formulation of Wilson loop operators in $\mathcal{N} = 4$, now in terms of an array of $D3$-branes.

5.2 Wilson loops as $D3$-branes

Let's now consider the $\mathcal{N} = 4$ gauge theory description of a configuration of multiple $D3$-branes in $\text{AdS}_5 \times \text{S}^5$. As we have argued in section 4, the only modification in the defect conformal field theory compared to the case with the $D5$-branes is to quantize the χ_i^I fields as chiral bosons as opposed to fermions. Therefore, we consider the defect conformal field theory action (5.3) treating χ_i^I now as bosons.

Similarly to the case with multiple $D5$-branes, we realize the charge induced by the fundamental strings ending on the $D3$-branes by the Wilson loop operator (5.4) in the representation $\rho = (k_1, \ldots, k_M)$ of $U(M)$, where $\rho = (k_1, \ldots, k_M)$ is a Young tableau of $U(M)$. A charge in the representation $\rho = (k_1, \ldots, k_M)$ is produced when k_i fundamental strings end on the i-th $D3$-brane. This $D3$-brane configuration can be labeled by the array $(D3_{k_1}, \ldots, D3_{k_M})$: Therefore, in order to integrate out the degrees of freedom on the

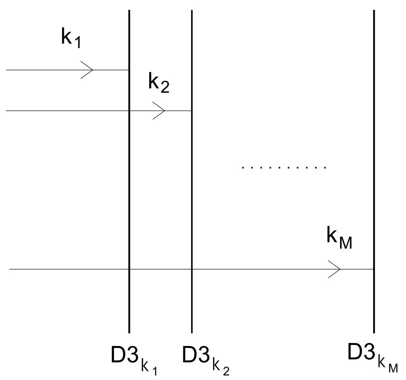

Figure 6: Array of strings producing a background charge given by the representation $\rho = (k_1, \ldots, k_M)$ of $U(M)$. The $D3$-branes are drawn separated for illustration purposes only, as they sit on top of each other.

probe $D3$-branes we must calculate the path integral (5.6) treating χ_i^I as bosons.

We gauge fix the $SU(N) \times U(M)$ as before. This gives us that χ_i^I are chiral bosons with energy $w_i + \Omega_I$. Their partition function is then given by

$$\prod_{i=1}^{N} \prod_{J=1}^{M} \left(\frac{1}{1 - x_i e^{i\beta\Omega_J}} \right), \tag{5.27}$$

where as before $x_i = e^{i\beta w_i}$ is an eigenvalue of the holonomy matrix appearing in the Wilson loop operator.

Taking into account the measure change computed earlier, we have that

$$Z = e^{iS_{\mathcal{N}=4}} \cdot \int [DU]\ \chi_{(k_1,\ldots,k_M)}(U^*) \prod_{i=1}^{N} \prod_{J=1}^{M} \left(\frac{1}{1 - x_i e^{i\beta\Omega_J}} \right), \tag{5.28}$$

where we have identified the operator insertion (5.4) with a character in the $\rho = (k_1, \ldots, k_M)$ representation of $U(M)$:

$$\chi_{(k_1,\ldots,k_M)}(U^*) \equiv \mathrm{Tr}_{(k_1,\ldots,k_M)} e^{-i\beta\tilde{A}_0}. \tag{5.29}$$

Now we use that the partition function of the bosons can be expanded in terms of characters of $U(M)$ by generalizing formula (4.19)

$$\begin{aligned} \prod_{J=1}^{M} \left(\frac{1}{1 - x_i e^{i\beta\Omega_J}} \right) &= \sum_{l=0}^{\infty} x_i^l\, \chi_{\,(l,0\ldots,0)} \\ (U) &= \sum_{l=0}^{\infty} x_i^l H_l(U_1, \ldots, U_M), \end{aligned} \tag{5.30}$$

where

$$H_l(U) = \mathrm{Tr}_{\,(l,0\ldots,0)}\, e^{i\beta\tilde{A}_0} \tag{5.31}$$

is the character of $U(M)$ in the l-th symmetric product representation.

Using an identity from [31]

$$\prod_{i=1}^{N} \sum_{l=0}^{\infty} x_i^l H_l(U) = \sum_{n_1 \geq n_2 \geq \ldots \geq n_N} \det(H_{n_j+i-j}(U))\ \chi_{(n_1,\ldots,n_N)}(x), \tag{5.32}$$

where

$$\chi_{(n_1,\ldots,n_N)}(x) = W_{(n_1,\ldots,n_N)} \tag{5.33}$$

is the half-BPS Wilson loop operator corresponding to the Young tableau $R = (n_1, \ldots, n_N)$ of $SU(N)$.

The Jacobi-Trudy identity (see e.g. [29]) implies that

$$\det(H_{n_j+i-j}(U)) = \chi_{(n_1,n_2,\ldots,n_N)}(U), \tag{5.34}$$

where $\chi_{(n_1,n_2,\ldots,n_N)}(U)$ is the character of $U(M)$ associated with the Young tableau $(n_1, n_2, \ldots, n_N)$. Considering the antisymmetry of the elements in the same column, we get the constraint that $n_{M+1} = \ldots = n_N = 0$, otherwise the character vanishes.

These computations allow us to write (5.28) as:

$$Z = e^{iS_{\mathcal{N}=4}} \cdot \sum_{n_1 \geq n_2 \geq \ldots \geq n_N} W_{(n_1,\ldots,n_N)} \cdot \int [DU]\ \chi_{(n_1,\ldots,n_N)}(U)\ \chi_{(k_1,\ldots,k_M)}(U^*). \tag{5.35}$$

Using

$$\int [DU]\ \chi_{(n_1,\ldots,n_N)}(U)\ \chi_{(k_1,\ldots,k_M)}(U^*) = \prod_{I=1}^{M} \delta_{n_I,k_I} \prod_{i=M+1}^{N} \delta_{n_i,0}, \tag{5.36}$$

we get that:

$$Z = e^{iS_{\mathcal{N}=4}} \cdot W_{(k_1,\ldots,k_M,\ldots,0)}. \tag{5.37}$$

We have shown that a collection of $D3$-branes described by the array $(D3_{k_1}, \ldots, D3_{k_M})$ in $\mathrm{AdS}_5\times\mathrm{S}^5$ corresponds to the half-BPS Wilson loop operator (2.3) in $\mathcal{N}=4$ SYM in the representation $R = (k_1, \ldots, k_N)$ of $SU(N)$

$$(D3_{k_1}, \ldots, D3_{k_M}) \longleftrightarrow Z = e^{iS_{\mathcal{N}=4}} \cdot W_{(k_1,\ldots,k_M,0,\ldots,0)}. \tag{5.38}$$

Therefore, any half-BPS Wilson loop operator in $\mathcal{N}=4$ has a bulk realization in terms of $D3$-branes.

To summarize, we have shown that a half-BPS Wilson loop described by an arbitrary Young tableau can be described in terms of a collection of $D5$-branes or $D3$-branes. We have shown that indeed the relation between a Wilson loop in an arbitrary representation and a D-brane configuration is precisely the one described in the introduction.

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Note added. While this paper was getting ready for publication, the preprint [32] appeared, which has overlap with parts of section 3. In [33] an analogous $D5$-brane solution was considered for the AdS black hole background.

A. Supersymmetry of Wilson loops in $\mathcal{N}=4$ SYM

In this appendix we study the constraints imposed by unbroken supersymmetry on the Wilson loop operators (2.1) of $\mathcal{N}=4$ SYM. Previous studies of supersymmetry of Wilson loops in $\mathcal{N}=4$ SYM include [34–36].

We want to impose that the Wilson loop operator (2.1) is invariant under one-half of the $\mathcal{N}=4$ Poincare supersymmetries and also invariant under one-half of the conformal supersymmetries. The Poincare supersymmetry transformations are given by

$$\begin{aligned} \delta_{\epsilon_1} A_\mu &= i\bar{\epsilon}_1\gamma_\mu\lambda \\ \delta_{\epsilon_1} \phi_I &= i\bar{\epsilon}_1\gamma_I\lambda, \end{aligned} \tag{A.1}$$

while the superconformal supersymmetry transformations are given

$$\begin{aligned}\delta_{\epsilon_2} A_\mu &= i\bar{\epsilon}_2 x^\nu \gamma_\nu \gamma_\mu \lambda \\ \delta_{\epsilon_2} \phi_I &= i\bar{\epsilon}_2 x^\nu \gamma_\nu \gamma_I \lambda,\end{aligned} \tag{A.2}$$

where $\epsilon_{1,2}$ are ten dimensional Majorana-Weyl spinors of opposite chirality. The use of ten dimensional spinors is useful when comparing with string theory computations.

Preservation of one-half of the Poincare supersymmetries locally at each point in the loop where the operator is defined yields:

$$P\epsilon_1 = (\gamma_\mu \dot{x}^\mu + \gamma_I \dot{y}^I)\epsilon_1 = 0. \tag{A.3}$$

Therefore, there are invariant spinors at each point in the loop if and only if $\dot{x}^2 + \dot{y}^2 = 0$. This requires that $x^\mu(s)$ is a time-like curve and that $\dot{y}^I = n^I(s)\sqrt{-\dot{x}^2}$, where $n^I(s)$ is a unit vector in R^6, satisfying $n^2(s) = 1$. Without loss of generality we can perform a boost and put the external particle labeling the loop at rest so that the curve along $R^{1,3}$ is given by $(x^0(s), x^i(s) = 0)$ and we can also choose an affine parameter s on the curve such that $\sqrt{-\dot{x}^2} = 1$.

In order for the Wilson loop to be supersymmetric, each point in the loop must preserve the same spinor. Therefore, we must impose that

$$\frac{dP(s)}{ds} = 0, \tag{A.4}$$

which implies that $\ddot{x}^0 = 0$ and that $n^I(s) = n^I$. Therefore, supersymmetry selects a preferred curve in superspace, the straight line Wilson loop operator, given by

$$W_R(C) = \mathrm{Tr}_R\, P \exp\left(i \int dt\, (A_0 + \phi)\right), \tag{A.5}$$

where $\phi = n^I \phi_I$. The operators are now just labelled by a choice of Young tableau R. For future reference, we write explicitly the 8 unbroken Poincare supersymmetries. They must satisfy

$$i\bar{\epsilon}_1 \gamma_0 \lambda + i n^I \bar{\epsilon}_1 \gamma_I \lambda = 0. \tag{A.6}$$

Using relations for conjugation of spinor with the conventions used here

$$\bar{\chi}\zeta = \bar{\zeta}\chi, \qquad \chi = \gamma^I \zeta \to \bar{\chi} = -\bar{\zeta}\gamma^I \tag{A.7}$$

we arrive at

$$\gamma_0 \gamma_I n^I \epsilon_1 = \epsilon_1. \tag{A.8}$$

In a similar manner it is possible to prove that the straight line Wilson loop operator (A.5) also preserves one-half of the superconformal supersymmetries. The 8 unbroken superconformal supersymmetries are given by:

$$\gamma_0 \gamma_I n^I \epsilon_2 = -\epsilon_2. \tag{A.9}$$

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B. Supersymmetry of Fundamental String and of $D5_k$-brane

In this appendix we show that the particular embeddings considered for the fundamental string and the $D5_k$-brane in section 3 preserve half of the supersymmetries of the background. We will use conventions similar to those in [37].

For convinience we write again the metric we are interested in (we set $L = 1$)

$$ds^2_{AdS\times S} = u^2 \eta_{\mu\nu} dx^\mu dx^\nu + \frac{du^2}{u^2} + d\theta^2 + \sin^2\theta \, d\Omega_4^2, \tag{B.1}$$

where the metric on S^4 is given by:

$$d\Omega_4 = d\varphi_1^2 + \sin\varphi_1^2 d\varphi_2^2 + \sin\varphi_1^2 \sin\varphi_2^2 d\varphi_3^2 + \sin\varphi_1^2 \sin\varphi_2^2 \sin\varphi_3^2 d\varphi_4^2. \tag{B.2}$$

It is useful to introduce tangent space gamma matrices, i.e. $\gamma_{\underline{m}} = e^m_{\underline{m}} \Gamma_m$ $(m, \underline{m} = 0, \ldots, 9)$ where $e^m_{\underline{m}}$ is the inverse vielbein and Γ_m are the target space matrices:

$$\begin{aligned}
&\gamma_\mu = \frac{1}{u}\Gamma_\mu \qquad (\mu = 0,1,2,3), \qquad \gamma_4 = u\Gamma_u, \qquad \gamma_5 = \Gamma_\theta, \\
&\gamma_{a+5} = \frac{1}{\sin\theta}\left(\prod_{j=1}^{a-1} \frac{1}{\sin\varphi_j}\right)\Gamma_{\varphi_a} \qquad (a = 1,2,3,4)
\end{aligned} \tag{B.3}$$

The Killing spinor of $AdS_5\times S^5$ in the coordinates (3.1) is given by [37]

$$\epsilon = \left[-u^{-\frac{1}{2}}\gamma_4 h(\theta,\varphi_a) + u^{\frac{1}{2}} h(\theta,\varphi_a)(\eta_{\mu\nu}x^\mu\gamma^\nu)\right]\eta_2 + u^{\frac{1}{2}} h(\theta,\varphi_a)\eta_1 \tag{B.4}$$

where

$$h(\theta,\varphi_a) = e^{\frac{1}{2}\theta\gamma_{45}} e^{\frac{1}{2}\varphi_1\gamma_{56}} e^{\frac{1}{2}\varphi_2\gamma_{67}} e^{\frac{1}{2}\varphi_3\gamma_{78}} e^{\frac{1}{2}\varphi_4\gamma_{89}} \tag{B.5}$$

η_1 and η_2 are constant ten dimensional complex spinors with negative and positive ten dimensional chirality, i.e.

$$\gamma_{11}\eta_1 = -\eta_1 \qquad \gamma_{11}\eta_2 = \eta_2. \tag{B.6}$$

They also satisfy:

$$P_-\eta_1 = \eta_1 \qquad P_+\eta_2 = \eta_2 \tag{B.7}$$

where $P_\pm = \frac{1}{2}(1 \pm i\gamma^{0123})$. Thus, each spinor $\eta_{1,2}$ has 16 independent real components. These can be written in terms of ten dimensional Majorana-Weyl spinors ϵ_1 and ϵ_2 of negative and positive chirality respectively:

$$\begin{aligned}
\eta_1 &= \epsilon_1 - i\gamma^{0123}\epsilon_1 \\
\eta_2 &= \epsilon_2 + i\gamma^{0123}\epsilon_2.
\end{aligned} \tag{B.8}$$

By going to the boundary of AdS at $u \to \infty$, we can identify from (B.4) ϵ_1 as the Poincare supersymmetry parameter while ϵ_2 is the superconformal supersymmetry parameter of $\mathcal{N} = 4$ SYM.

The supersymmetries preserved by the embedding of a probe, are those that satisfy

$$\Gamma_\kappa\epsilon = \epsilon \tag{B.9}$$

where Γ_κ is the κ symmetry transformation matrix in the probe worldvolume theory and ϵ is the Killing spinor of the $AdS_5 \times S_5$ background (B.4). Both Γ_κ and ϵ have to be evaluated at the location of the probe.

Let's now consider a fundamental string with an AdS_2 worldvolume geometry with embedding:

$$\sigma^0 = x^0 \qquad \sigma^1 = u \qquad x^i = 0 \qquad x^I = n^I. \tag{B.10}$$

The position of the string on the S^5 is parametrized by the five constant angles $(\theta, \varphi_1, \varphi_2, \varphi_3, \varphi_4)$ or alternatively by a unit vector n^I in R^6. The matrix Γ_κ for a fundamental string with this embedding reduces to

$$\Gamma_{F1} = \gamma_{04} K \tag{B.11}$$

where K acts on a spinor ψ by $K\psi = \psi^*$. For later convenience we define also the operator I such that $I\psi = -i\psi$.

The equation (B.9) has to be satisfied at every point on the string. Thus, the term proportional to $u^{\frac{1}{2}}$ gives:

$$\Gamma_{F1} h(\theta, \varphi_a)\eta_1 = h(\theta, \varphi_a)\eta_1. \tag{B.12}$$

The terms proportional to $u^{-\frac{1}{2}}$ and $u^{-\frac{1}{2}} x_0$ both give:

$$\Gamma_{F1} h(\theta, \varphi_a)\eta_2 = -h(\theta, \varphi_a)\eta_2. \tag{B.13}$$

These can be rewritten as

$$n^I \gamma_{0I}\eta_1 = \eta_1^* \qquad n^I \gamma_{0I}\eta_2 = -\eta_2^* \qquad I = 4, 5, 6, 7, 8, 9 \tag{B.14}$$

where

$$n^I(\theta, \varphi_1, \varphi_2, \varphi_3, \varphi_4) = \begin{pmatrix} \cos\theta \\ \sin\theta\cos\varphi_1 \\ \sin\theta\sin\varphi_1\cos\varphi_2 \\ \sin\theta\sin\varphi_1\sin\varphi_2\cos\varphi_3 \\ \sin\theta\sin\varphi_1\sin\varphi_2\sin\varphi_3\cos\varphi_4 \\ \sin\theta\sin\varphi_1\sin\varphi_2\sin\varphi_3\sin\varphi_4 \end{pmatrix} = \begin{pmatrix} \cos\theta \\ \sin\theta v^\alpha \end{pmatrix}, \tag{B.15}$$

where $\alpha = (5, 6, 7, 8, 9)$ and these vectors satisfy $n^2 = 1$ and $v^2 = 1$. Considering the parametrization (B.8), the projection (B.14) becomes:

$$\gamma_{0I} n^I \epsilon_1 = \epsilon_1 \qquad \gamma_{0I} n^I \epsilon_2 = -\epsilon_2. \tag{B.16}$$

We note that n^I define the position of the string in the S_5, so it characterizes the unbroken rotational symmetry of the system. Therefore, the fundamental string preserves exactly the same supersymmetries as the Wilson loop operator (A.5).

We now study the $D5_k$-brane embedding considered first by [20, 21]:

$$\sigma^0 = x^0 \qquad \sigma^1 = u \qquad \sigma^a = \varphi_a \qquad x^i = 0 \qquad \theta = \theta_k = \text{constant}. \tag{B.17}$$

There is an electric flux on the brane given by

$$F_{04} = F = \cos\theta_k, \tag{B.18}$$

where k is the amount of fundamental string charge on the $D5_k$-brane.

For this configuration, Γ_κ is

$$\begin{aligned}\Gamma_{D5} &= \frac{1}{\sqrt{1-F^2}}\gamma_{046789}KI + \frac{F}{\sqrt{1-F^2}}\gamma_{6789}I \\ &= \frac{1}{\sin\theta_k}\gamma_{046789}KI + \frac{\cos\theta_k}{\sin\theta_k}\gamma_{6789}I\end{aligned} \tag{B.19}$$

Following similar steps as for the fundamental string, we arrive at

$$\Gamma_{D5}h(\theta_k,\varphi_a)\epsilon_1 = h(\theta_k,\varphi_a)\epsilon_1 \qquad \bar{\Gamma}_{D5}h(\theta_k,\varphi_a)\epsilon_2 = h(\theta_k,\varphi_a)\epsilon_2, \tag{B.20}$$

where

$$\bar{\Gamma}_{D5} = -\frac{1}{\sin\theta_k}\gamma_{046789}KI + \frac{\cos\theta_k}{\sin\theta_k}\gamma_{6789}I. \tag{B.21}$$

Using that $h^{-1}\gamma_{04}h = n^I\gamma_{0I}$ and that $h^{-1}\gamma_{6789}h = l^\alpha\gamma_{\alpha 56789}$ we have that the supersymmetry left unbroken by a $D5_k$-brane is given by:

$$\gamma_{04}\epsilon_1 = \epsilon_1 \qquad \gamma_{04}\epsilon_2 = -\epsilon_2. \tag{B.22}$$

Therefore it preserves the same supersymmetries as a fundamental string sitting at the north pole (i.e. $\theta = 0$), labeled by the vector $n^I = (1,0,0,0,0,0)$. This vector selects the unbroken rotational symmetry.

C. Gauge fixing and the unitary matrix measure

In section 5 we have gauge fixed the $U(M)$ symmetry by imposing the diagonal, constant gauge:

$$\tilde{A}_0 = \text{diag}(\Omega_1,\ldots,\Omega_M). \tag{C.1}$$

There is an associated Fadeev-Popov determinant Δ_{FP} corresponding to this gauge choice. This modifies the measure to

$$[D\tilde{A}_0]\cdot\Delta_{FP}, \tag{C.2}$$

where now $[D\tilde{A}_0]$ involves integration only over the constant mode of the hermitean matrix $\tilde{A}_0$. Under an infinitessimal gauge transformation labelled by α, $\tilde{A}_0$ transforms by

$$\delta\tilde{A}_0 = \partial_t\alpha + i[\tilde{A}_0,\alpha], \tag{C.3}$$

so that:

$$\Delta_{FP} = \det\left(\partial_t \ + i[\tilde{A}_0,\ ]\right). \tag{C.4}$$

An elementary computation yields

$$\Delta_{FP} = \prod_{l\neq 0}^{\infty}\frac{2\pi i l}{\beta}\prod_{I<J}\prod_{k=1}^{\infty}\left(1 - \frac{\beta^2(\Omega_I-\Omega_J)^2}{4\pi^2k^2}\right), \tag{C.5}$$

where we have introduced β as an infrared regulator. Now, using the product representation of the *sin* function we have that up to an irrelevant constant:

$$\Delta_{FP} = \prod_{I<J} 4\, \frac{\sin^2\left(\beta\left(\frac{\Omega_I - \Omega_J}{2}\right)\right)}{(\Omega_I - \Omega_J)^2}. \tag{C.6}$$

This together with the formula for the measure of the Hermitean matrix $\tilde{A}_0$

$$[D\tilde{A}_0] = \prod_{I<J} d\Omega_I (\Omega_I - \Omega_J)^2 \tag{C.7}$$

proves our claim that the gauge fixing effectively replaces the measure over the Hermitean matrix $\tilde{A}_0$ by the measure over the unitary $U = e^{i\beta \tilde{A}_0}$

$$[D\tilde{A}_0] \cdot \Delta_{FP} = [DU] = \prod_{I<J} d\Omega_I \Delta(\Omega) \bar{\Delta}(\Omega), \tag{C.8}$$

where

$$\Delta(\Omega) = \prod_{I<J} (e^{i\beta\Omega_I} - e^{i\beta\Omega_J}). \tag{C.9}$$

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